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# SENSITIVITY ANALYSIS IN NETWORKS

by  
Richard Wollmer

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# **SENSITIVITY ANALYSIS OF NETWORKS**

by

**Richard Wollmer**

**Operations Research Center  
University of California, Berkeley**

**April 1965**

**ORC 65-8**

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## TABLE OF CONTENTS

### CHAPTER I

Introduction.....	1
-------------------	---

### CHAPTER II

Basic Concepts.....	3
---------------------	---

1. Max-Flow Min-Cut Theorem.....	3
----------------------------------	---

2. The Topological Dual.....	3
------------------------------	---

### CHAPTER III

Deterministic Case.....	7
-------------------------	---

1. Problem Formulation.....	7
-----------------------------	---

2. The Algorithm.....	8
-----------------------	---

3. Justification of the Algorithm.....	9
----------------------------------------	---

### CHAPTER IV

Stochastic Case.....	16
----------------------	----

1. Problem Formulation.....	16
-----------------------------	----

2. The Algorithm.....	18
-----------------------	----

3. Justification of the Algorithm.....	21
----------------------------------------	----

4. Normal Case.....	35
---------------------	----

5. Validity of the Bound.....	36
-------------------------------	----

6. Practicality of the Algorithm.....	43
---------------------------------------	----

### CHAPTER V

Other Approaches.....	45
-----------------------	----

Appendix I. Example for Deterministic Case.....	48
-------------------------------------------------	----

Appendix II. Example for Stochastic Case.....	51
-----------------------------------------------	----

List of Symbols.....	56
----------------------	----

References.....	57
-----------------	----

## CHAPTER I

### Introduction

A maximum flow network is defined by a set of arcs and a set of points called nodes. Each arc joins two nodes and has associated with it a positive capacity which represents the maximum amount of flow that may pass over it. One of the nodes is designated as the source and another as the sink. From these nodes, arc, and capacities the maximum amount of flow that may pass from source to sink may be calculated.

This investigation is concerned with a sensitivity analysis on such networks. Specifically, each arc of the network is assumed to be subject to breakdowns which result in a reduction in its capacity. The problem is to find the greatest reduction in maximum flow possible if  $n$  breakdowns occur and to find where these breakdowns must occur for this reduction to result. The methods developed in this report solve this problem for a certain class of networks known as planar networks.

The first method solves the problem exactly where it is assumed that each arc is subject to only one breakdown and the amount by which the capacity of an arc is reduced due to a breakdown is a deterministic quantity. The algorithm developed could, however, easily be modified to allow for multiple breakdowns on one arc.

The second method solves this problem approximately when the amount by which the capacity of an arc is reduced is a random

variable with unknown distribution but with known mean and variance. For this case, multiple breakdowns are allowed for the individual arcs. The algorithm developed can also be used to solve the problem more exactly when these random variables have normal distributions.

## CHAPTER II

### Basic Concepts

#### 1. Max-Flow Min-Cut Theorem

The central idea in the solving of maximum flow network problems is summarized in the max-flow min-cut theorem

Definition: Consider a network consisting of nodes which include a source and a sink, and capacitated arcs which join two nodes. Let  $A$  and  $B$  be a partition of the nodes such that the source is in  $A$  and the sink is in  $B$ . Then the set of arcs which join a node in  $A$  to a node in  $B$  is called a cut set and is denoted  $[A, B]$ . Furthermore, the value of this cut set,  $V[A, B]$ , is equal to the sum of the capacities of its arcs.

A property of any cut set,  $[A, B]$ , is that any path from source to sink must use at least one of its arcs. Thus, it is apparent that the maximum flow cannot exceed the minimum value of all cut sets. The max-flow min-cut theorem states that the maximum flow actually equals the minimum cut. <sup>(1)</sup>

#### 2. The Topological Dual

The topological dual of a network, when defined, is another network in which the arcs, instead of having capacities, have lengths. Furthermore, there is a one-to-one correspondence between the proper cuts of the original network and the routes through the dual, and the problem of finding the minimum cut may be re-

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<sup>(1)</sup>Reference 8.

duced to one of finding a shortest route.

Let the original maximum flow network be called the primal. To the primal add an artificial arc extending from the source to the sink and having a capacity of zero. The resulting network will be referred to as the modified primal. The dual network is defined if and only if the primal is source-sink planar, a source-sink planar network being one where the modified primal can be drawn on a sphere in such a way that no two arcs intersect except at a node.

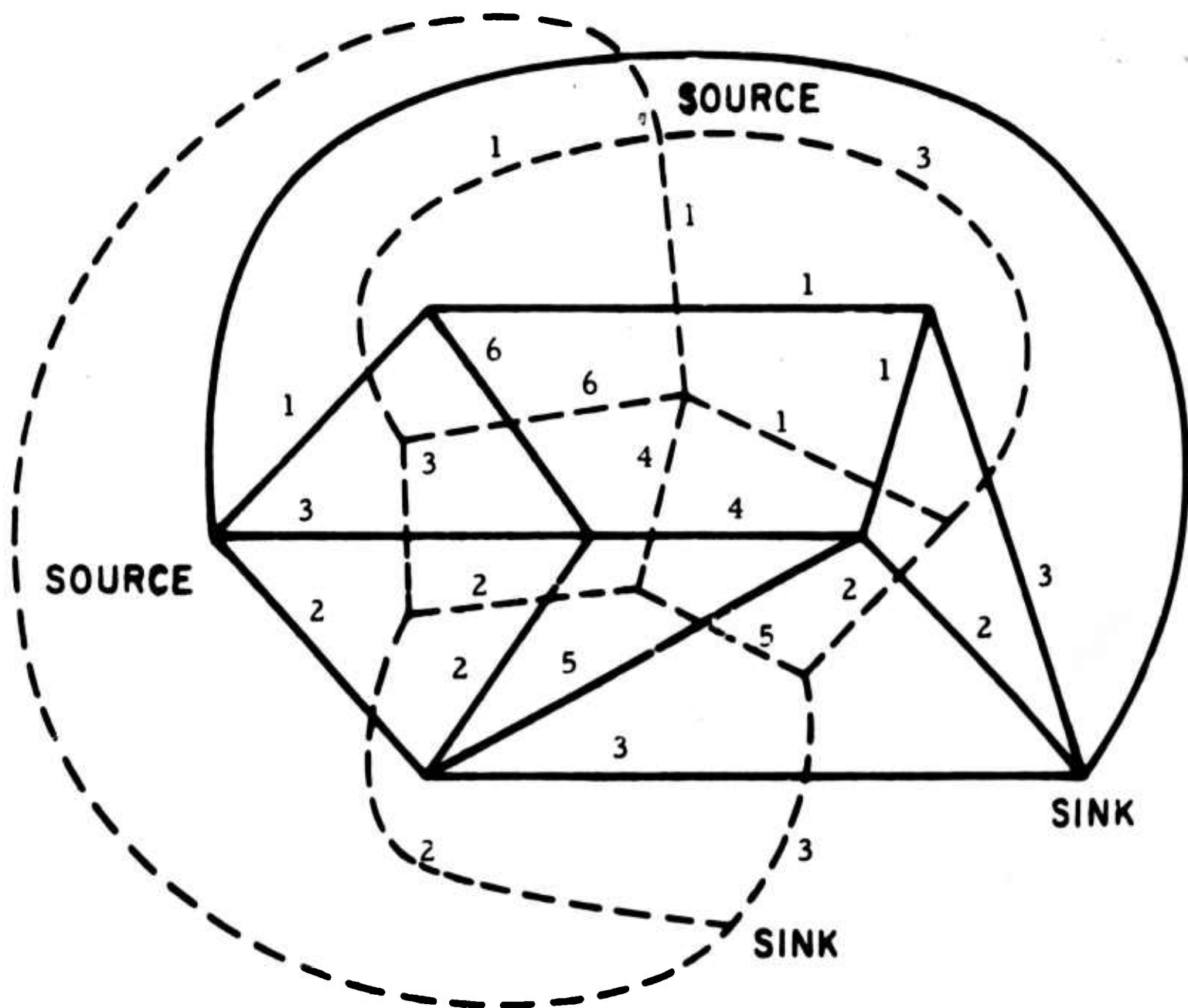
When defined the dual is constructed in the following manner:

1. Draw the modified primal on a sphere in such a way that no two arcs intersect except at a node.
2. Place a node in each mesh of the modified primal. Let the node in one of the two meshes bounded by the artificial arc be the source and the node in the other of these two meshes be the sink.
3. For each arc except the artificial one construct an arc of the dual that intersects it and joins the nodes in the meshes on either side of it.
4. Assign each arc of the dual a length equal to the capacity of the primal arc it intersects.

An example of a network and its dual is shown in Figure 1. Letting a route through the dual be any path from its source to its sink, it follows that there is a one-to-one correspondence between the proper cuts of the primal and the routes of the dual.<sup>(1)</sup> Specifically if  $A$  is any route through the dual then the arcs of the primal

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<sup>(1)</sup>Reference 8.



$\text{---}\overset{a}{\text{---}}$  ARC OF PRIMAL WITH CAPACITY  $a$   
 $\text{---}\overset{b}{\text{---}}$  ARC OF DUAL WITH LENGTH  $b$

**FIG. 1 A NETWORK AND ITS DUAL**

which intersect  $A$  form a proper cut and conversely if  $B$  is a proper cut of the primal, then the arcs of the dual which intersect  $B$  form a route.

## **CHAPTER III**

### **Deterministic Case**

#### **1. Problem Formulation**

The problem solved in this chapter is the following:

A source-sink planar maximum flow network is given in which each arc is subject to, at most, one breakdown: a breakdown resulting in a reduction in the capacity of that arc by a known quantity. It is desired to find the greatest reduction in maximum flow possible from, at most,  $n$  breakdowns and where these breakdowns must occur, to give this reduction.

Since the maximum flow of a network is equal to its minimum cut which in turn is equal to the length of the shortest route through its dual, it follows that solving this problem is equivalent to solving the following problem for the dual:

A network is given in which the arcs are assigned lengths. Each arc is subject to an improvement which results in a decrease in its length by a known quantity. It is desired to find the smallest value possible for the length of the shortest route, if no more than  $n$  improvements occur and where these improvements must occur in order to achieve this.

It is the latter problem that the algorithm of this chapter solves. Throughout the rest of this chapter all nodes and arcs refer to the dual network unless specified otherwise.

## 2. The Algorithm

Let an  $i$ -arc path to node  $a$  be any path from the source to node  $a$  along which at most  $i$  improvements occur. Also, let  $L_{a,i}$  (to be determined later) be the length of the shortest  $i$ -arc path to node  $a$ . The algorithm assigns to each node  $a$ ,  $n+1$  labels,  $(D,t,k)_{a,0}$ ,  $(D,t,k)_{a,1}$ ,  $\dots$ ,  $(D,t,k)_{a,n}$ . The individual components of  $(D,t,k)_{a,i}$  will be denoted by  $D_{a,i}$ ,  $t_{a,i}$  and  $k_{a,i}$ . Initially, it is known that all  $D_{a,i} \geq L_{a,i}$ . The  $D_{a,i}$  are then decreased in such a way as to preserve this initial property until all  $D_{a,i} = L_{a,i}$ . At this point  $D_{\bar{S},n}$  is the length of the desired path where  $\bar{S}$  is the sink. The components  $t_{a,i}$  and  $k_{a,i}$  are tracers which are used to find the desired path itself.

Let  $l(a,b)$  be the length of arc  $(a,b)$ ,  $d(a,b)$  the decrease in this length due to an improvement,  $S$  the source, and  $\bar{S}$  the sink. Note that  $l(a,b) = l(b,a)$  and  $d(a,b) = d(b,a)$ . The algorithm for finding the length of the desired route is as follows:

1. For  $i = 0, 1, \dots, n$  set  $D_{\bar{S},i} = 0$  and  $D_{a,i} = \infty$  for  $a \neq \bar{S}$ . Set  $m = 0$ .
2. Check each arc  $(a,b)$  and:
  - a. If  $D_{a,m} > D_{b,m} + l(a,b)$  set
 
$$D_{a,m} = D_{b,m} + l(a,b)$$

$$t_{a,m} = b$$

$$k_{a,m} = 0$$

b. If  $m \geq 1$  and  $D_{a,m} > D_{b,m-1} + l(a,b) - d(a,b)$   
 set  
 set

$$D_{a,m} = D_{b,m-1} + l(a,b) - d(a,b)$$

$$t_{a,m} = b$$

$$k_{a,m} = 1$$

3. Repeat 2 until no more changes can be made. Then if  $m < n$ , increase  $m$  by 1 and go back to 2. If  $m = n$ , terminate, as  $D_{\bar{S},n}$  is the length of the desired route.

The desired route itself may be found by the following procedure:

1. Set  $m = 1$ ,  $\bar{S} = a_1$ ,  $i_1 = n$ .

2. Let  $a_{m+1} = t_{a_m, i_m}$  and  $i_{m+1} = i_m - k_{a_m, i_m}$ .

3. If  $a_{m+1} \neq \bar{S}$ , increase  $m$  by 1 and go back to 2.

Otherwise terminate.

$S = a_{m+1}, \dots, a_1 = \bar{S}$  is the desired path. The improvements must occur on those arcs  $(a_{j+1}, a_j)$  where  $k_{a_j, i_j} = 1$ . The arcs in the primal on which the breakdowns must occur are the arcs which intersect these arcs of the dual.

### 3. Justification of the Algorithm

The procedure for justifying the algorithm of the last section will be to show first that all  $D_{a,i} \geq L_{a,i}$  always and  $D_{a,i} = L_{a,i}$  at termination. Then a relationship between the  $L_{a,i}$  will be established to help verify the process of tracing out the desired path.

Lemma 1:  $D_{a,i} \geq L_{a,i}$  all  $a$  and all  $i$ .

Proof: Assume that at one point  $D_{a,i} \geq L_{a,i}$  all  $a$  and all  $i$ . Suppose that some  $D_{a,i}$  is changed to  $\bar{D}_{a,i}$ . Then there is a node  $b$  such that either:

1.  $\bar{D}_{a,i} = D_{b,i} + l(a,b)$  or
2.  $\bar{D}_{a,i} = D_{b,i-1} + l(a,b) - d(a,b)$ .

In the first case any  $i$ -arc path to node  $b$  of length  $L_{b,i}$  plus arc  $(a,b)$  at its true length of  $l(a,b)$  is an  $i$ -arc path of length less than or equal to  $\bar{D}_{a,i}$ . In the second case any  $(i-1)$ -arc path to node  $b$  of length  $L_{b,i-1}$  plus arc  $(a,b)$  at its improvement length of  $l(a,b) - d(a,b)$  is an  $i$ -arc path to node  $a$  of length less than or equal to  $\bar{D}_{a,i}$ . Thus the relationship  $D_{a,i} \geq L_{a,i}$  all  $a$  and all  $i$  still holds. Initially,  $D_{S,i} = 0 = L_{S,i}$  and  $D_{a,i} = \infty > L_{a,i}$  for  $a \neq S$ . The lemma follows from induction.

Lemma 2: After a finite number of iterations,  $D_{a,0} = L_{a,0}$  for all  $a$  and remains at that value for all subsequent iterations.

Proof: Let  $S, a_1, \dots, a_m, a$  be a shortest route from  $S$  to  $a$ . After one examination of the arcs,

$$D_{a_1,0} \leq l(S, a_1)$$

After 2 iterations,

$$D_{a_2,0} \leq l(S, a_1) + l(a_1, a_2)$$

After  $m+1$  iterations,

$$D_{a,0} \leq l(S, a_1) + \sum_{i=1}^{m-1} l(a_i, a_{i+1}) + l(a_m, a) = L_{a,0}$$

Of course  $D_{a,0}$  will remain at  $L_{a,0}$  for all subsequent iterations since the  $D_{a,0}$  are non-increasing and  $D_{a,0} \geq L_{a,0}$  by lemma 1. Thus the lemma holds for any particular node. Let  $m(a)$  be the number of iterations required in order that  $D_{a,0} = L_{a,0}$ . Then after  $\max_a m(a)$  iterations  $D_{a,0} = L_{a,0}$  for all  $a$ .

Lemma 3: Suppose  $S, a_1, \dots, a_m$  is a shortest  $i$ -arc path from  $S$  to  $a_m$ . If arc  $(a_{m-1}, a_m)$  has an improvement in this path then

$$1. \quad L_{a_m, i} = L_{a_{m-1}, i-1} + l(a_{m-1}, a_m) - d(a_{m-1}, a_m)$$

Otherwise

$$2. \quad L_{a_m, i} = L_{a_{m-1}, i} + l(a_{m-1}, a_m)$$

Proof: Suppose arc  $(a_{m-1}, a_m)$  has an improvement in this path. Then  $S, a_1, \dots, a_{m-1}$  is an  $(i-1)$ -arc path to  $a_{m-1}$  of length  $L_{a_m, i} - l(a_{m-1}, a_m) + d(a_{m-1}, a_m)$  and

$$L_{a_{m-1}, i} \leq L_{a_m, i} - l(a_{m-1}, a_m) + d(a_{m-1}, a_m) \quad \text{or}$$

$$L_{a_m, i} \geq L_{a_{m-1}, i} + l(a_{m-1}, a_m) - d(a_{m-1}, a_m)$$

Furthermore, any  $(i-1)$ -arc path of length  $L_{a_{m-1}, i-1}$  to  $a_{m-1}$  and arc  $(a_{m-1}, a_m)$  at its improvement length is an  $i$ -arc path to  $a_m$  of length  $L_{a_{m-1}, i-1} + l(a_{m-1}, a_m) - d(a_{m-1}, a_m)$  and the first equality holds. If arc  $(a_{m-1}, a_m)$  does not have an improvement, then  $S, a_1, \dots, a_{m-1}$  is an  $i$ -arc path to  $a_{m-1}$  of length

$$L_{a_m, i} - l(a_{m-1}, a_m) \text{ and } L_{a_m, i} \geq L_{a_{m-1}, i} + l(a_{m-1}, a_m) :$$

Also, any  $i$ -arc path to  $a_{m-1}$  of length  $L_{a_{m-1}, i}$  and arc  $(a_{m-1}, a_m)$  is an  $i$ -arc path of length  $L_{a_{m-1}, i} + l(a_{m-1}, a_m)$  and the second equality holds.

Lemma 4: Suppose  $S, a_1, \dots, a_m$  is the shortest  $i$ -arc path from  $S$  to  $a_m$ . Let  $(a_r, a_{r+1})$  be the last arc in this path to have an improvement. Then, if  $i \geq 1$  :

$$L_{a_m, i} = L_{a_r, i-1} - d(a_r, a_{r+1}) + \sum_{j=r}^{m-1} l(a_j, a_{j+1})$$

Proof:

$$L_{a_{j+1}, i} - L_{a_j, i} = l(a_j, a_{j+1}) \quad j = r+1, \dots, m-1$$

$$L_{a_{r+1}, i} - L_{a_r, i-1} = l(a_r, a_{r+1}) - d(a_r, a_{r+1})$$

from Lemma 3. Summing these equations, gives

$$L_{a_m, i} = L_{a_r, i-1} - d(a_r, a_{r+1}) + \sum_{j=r}^{m-1} l(a_j, a_{j+1})$$

Theorem 5: The algorithm terminates after a finite number of iterations with  $D_{a, i} = L_{a, i}$  all  $a$  and all  $i$ .

Proof: Suppose that after a finite number of iterations  $D_{a, i} = L_{a, i}$  all  $a$  and all  $i \leq M$ . Let  $S, a_1, \dots, a_m$  be an  $(M+1)$ -arc path of length  $L_{a_m, M+1}$  and let  $(a_r, a_{r+1})$  be the last arc in this path in which an improvement occurs. Then, after one additional iteration,

$$D_{a_{r+1}, M+1} \leq L_{a_r, M} - d(a_r, a_{r+1}) + l(a_r, a_{r+1})$$

and after  $m - r$  additional iterations,

$$D_{a_m, M+1} \leq L_{a_r, M} - d(a_r, a_{r+1}) + \sum_{j=r}^{m-1} l(a_j, a_{j+1}) = L_{a_m, M+1}$$

and the theorem holds for any particular  $D_{a, M+1}$ . Let  $m(a)$  be the number of iterations required in order that  $D_{a, M+1} = L_{a, M+1}$ . Then after  $\max_a m(a)$  iterations  $D_{a, M+1} = L_{a, M+1}$  all  $a$ . Since  $D_{a, 0} = L_{a, 0}$  after a finite number of iterations, the theorem follows from induction.

Of course, any route from source to sink with  $n$  or fewer improvements is an  $n$ -arc path to  $\bar{S}$ . Thus at termination  $D_{\bar{S}, n} = L_{\bar{S}, n}$  is the length of the desired route. It now remains to justify the procedure for tracing out the desired path itself.

Lemma 6: If  $k_{a_j, i_j} = 1$ , then

$$L_{a_j, i_j} = L_{a_{j+1}, i_{j+1}} + l(a_j, a_{j+1}) - d(a_j, a_{j+1})$$

and if  $k_{a_j, i_j} = 0$  then

$$L_{a_j, i_j} = L_{a_{j+1}, i_{j+1}} + l(a_j, a_{j+1}).$$

Proof: Suppose  $k_{a_j, i_j} = 1$ .

Then

$$L_{a_j, i_j} = D_{a_{j+1}, i_{j+1}} + l(a_j, a_{j+1}) - d(a_j, a_{j+1})$$

held when the label  $L_{a_j, i_j}$  was assigned. Furthermore,

$D_{a_{j+1}, i_{j+1}} = L_{a_{j+1}, i_{j+1}}$  at this time for if not any  $i_{j+1} = (i_j - 1)$ -arc

path to node  $a_{j+1}$  of length  $L_{a_{j+1}, i_{j+1}}$  plus arc  $(a_j, a_{j+1})$  at its improvement length is an  $i_j$ -arc path to node  $a_j$  of length less than  $L_{a_j, i_j}$ . Also, if  $k_{a_j, i_j} = 0$ , then

$$L_{a_j, i_j} = D_{a_{j+1}, i_{j+1}} + l(a_j, a_{j+1})$$

held when the label  $L_{a_j, i_j}$  was assigned. Furthermore

$D_{a_{j+1}, i_{j+1}} = L_{a_{j+1}, i_{j+1}}$  held for if not any  $i_{j+1} = i_j$ -arc path to node  $a_{j+1}$  plus arc  $(a_j, a_{j+1})$  at its true length of  $l(a_j, a_{j+1})$  is an  $i_j$ -arc path to node  $a_j$  of length less than  $L_{a_j, i_j}$ .

**Lemma 7:** The procedure for tracing the desired path is finite. (i. e., there exists  $r$  such that  $a(r) = S$ .)

**Proof:**  $L_{a_{j+1}, i_{j+1}} \leq L_{a_j, i_j}$  and  $i_{j+1} \leq i_j$

with strict inequality for at least one of these. Thus no pair  $(a_r, i_r)$  can be repeated and the procedure must be finite.

**Theorem 8:** Let  $a_1, \dots, a_{m+1}$  be the nodes found by the tracing procedure. Then  $S = a_{m+1}, \dots, a_1 = \bar{S}$  with an improvement occurring on arc  $(a_j, a_{j+1})$  if and only if  $k_{a_j, i_j} = 1$  is the shortest  $n$ -arc path to  $\bar{S}$ .

**Proof:** Suppose  $S = a_{m+1}, \dots, a_r$  with an improvement on arc  $(a_j, a_{j+1})$  if and only if  $k_{a_j, i_j} = 1$  is an  $i_r$ -arc path to  $a_r$  of length  $L_{a_r, i_r}$ . Then:

$$1. L_{a_{r-1}, i_{r-1}} = L_{a_r, i_r} + l(a_{r-1}, a_r) \quad \text{if } k_{a_{r-1}, i_{r-1}} = 0$$

or

$$2. L_{a_{r-1}, i_{r-1}} = L_{a_r, i_r} + l(a_{r-1}, a_r) - d(a_{r-1}, a_r) \quad \text{if}$$

$$k_{a_{r-1}, i_{r-1}} = 1$$

In either case  $a_{m+1}, \dots, a_{r-1}$  with an improvement on arc  $(a_j, a_{j+1})$  if and only if  $k_{a_j, i_j} = 1$  is an  $i_{r-1}$ -arc path to  $a_{r-1}$  of length  $L_{a_{r-1}, i_{r-1}}$ . Since  $a_{m+1} = S$  is a path of zero length and hence an  $i_{m+1}$ -arc path to  $a_{m+1}$  of length  $L_{a_{m+1}, i_{m+1}}$  it follows from induction that  $a_{m+1}, \dots, a_1$  is an  $n$ -arc path to  $a_1$  of length  $L_{a_1, i_1} = L_{S, n}$ .

This completes the justification of the procedure for finding the desired route itself. The arcs where improvements must occur are those arcs  $(a_j, a_{j+1})$  where  $k_{a_j, i_j} = 1$ . The arcs of the primal upon which breakdowns must occur in order to minimize the maximum flow are those which intersect these arcs of the dual.

## CHAPTER IV

### Stochastic Case

#### 1. Problem Formulation

The arcs of the maximum flow network in this chapter are subject to anywhere from 0 to  $n$  breakdowns. The decrease in capacity of arc  $(a,b)$  due to  $i$  breakdowns is a random variable with unknown distribution but with known mean and variance. The mean and variance of this quantity will be denoted as  $\mu(a,b,i)$  and  $\sigma^2(a,b,i)$  respectively. Furthermore, these distributions are independent for the different arcs. It is desired to find the smallest possible value of  $F$  resulting from, at most,  $n$  breakdowns where  $F$  satisfies:

$$P \{ \text{max flow} \geq F \} \leq \beta$$

where  $P$  stands for probability and  $\beta$  is a small number between zero and one.

This problem can also be formulated in terms of the dual network. Specifically, each arc of the dual is subject to anywhere from 0 to  $n$  improvements. The decrease in length of arc  $(a,b)$  due to  $i$  improvements is a random variable with mean  $\mu(a,b,i)$  and variance  $\sigma^2(a,b,i)$ . It is desired to find the smallest possible value of  $F$  resulting from, at most,  $n$  total improvements on  $n$  or less arcs where  $F$  satisfies:

$$P \{ \text{min route} \geq F \} \leq \beta$$

It is the latter formulation that is considered in this chapter.

Since examples can be constructed to show that different distributions with the same  $\mu(a, b, i)$  and  $\sigma^2(a, b, i)$  can have different  $F$  and different locations for the improvements, the information given is not sufficient to determine a solution. Accordingly, the distribution considered will be the one whose smallest value of  $F$  is maximum.

Suppose that if improvements occur on certain arcs, the length of a particular route is a random variable  $L$  with mean  $\mu$  and variance  $\sigma^2$ . It follows from Tchebysheff's extended lemma<sup>(1)</sup>

$$P \{L - \mu \geq \epsilon\} \leq \frac{\sigma^2}{\epsilon^2 + \sigma^2}$$

or

$$P \{L \geq \mu + \epsilon\} \leq \frac{\sigma^2}{\epsilon^2 + \sigma^2}$$

If it is required to minimize  $\mu + \epsilon$  under the condition that:

$$\frac{\sigma^2}{\epsilon^2 + \sigma^2} \leq \beta$$

it follows that

$$\epsilon = \sigma \sqrt{\frac{1}{\beta} - 1}$$

and one obtains

$$P \{L \geq \mu + \sigma \sqrt{\frac{1}{\beta} - 1}\} \leq \beta$$

Thus the quantity  $\mu + \sigma \sqrt{\frac{1}{\beta} - 1}$  plays the role of the length

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<sup>(1)</sup>Reference 9, pp. 111-126.

of the above mentioned route and will be referred to as its effective length. The algorithm to be presented in this chapter finds those arcs upon which the  $n$  improvements must occur in order that the minimum effective length of all routes through the dual be minimized.

In order that this be a valid criteria the following assumptions are made:

1.  $\mu(a, b, i) < l(a, b)$  and is strictly increasing in  $i$ .
2.  $\mu(a, b, i) - \sigma(a, b, i) \sqrt{\frac{1}{\beta} - 1} > 0$  for  $i \geq 1$ .
3.  $\mu(a, b, i) > \mu(\bar{a}, \bar{b}, \bar{i})$  implies  

$$\mu(a, b, i) - \sigma(a, b, i) \sqrt{\frac{1}{\beta} - 1} > \mu(\bar{a}, \bar{b}, \bar{i}) - \sigma(\bar{a}, \bar{b}, \bar{i}) \times \sqrt{\frac{1}{\beta} - 1}$$

These assumptions assure that the solution to this problem involves exactly  $n$  breakdowns, assure that the effective length of an arc is always greater than zero and generate arguments in favor of the efficiency of the algorithm.

## 2. The Algorithm

Let an  $i$ -arc path to node  $a$  be any path from the source to node  $a$  with  $i$  or fewer improvements occurring on its arcs. Each node is assigned  $n + 1$  sets of labels, the set numbers being designated as  $0, 1, \dots, n$ . Each label consists of four components and is denoted  $(\mu, \sigma^2, t, k)_{a,i}^j$  where  $a$  designates the node,  $i$  the set number and  $j$  the rank within the set. The individual components are denoted as  $\mu_{a,i}^j$ ,  $(\sigma^2)_{a,i}^j$ ,  $t_{a,i}^j$ , and  $k_{a,i}^j$ . The quantities  $\mu_{a,i}^j$  and  $(\sigma^2)_{a,i}^j$  are related to the mean and variance of  $i$ -arc paths to node  $a$  and at termination  $\mu_{S,n}^J$  and

$(\sigma^2)_{S,n}^j$  are the mean and variance of the length of the desired path where  $j = \max j$  in this set. The components  $t_{a,i}^j$  and  $k_{a,i}^j$  are tracers which are used to find the path itself. Again,  $S$  and  $\bar{S}$  refer to the source and sink respectively.

The algorithm for finding the mean and variance of the length of the desired route is as follows:

1. Set  $i = 0$

2. Set  $\mu_{S,i}^1 = (\sigma^2)_{S,i}^1 = 0$ ,  $\mu_{a,i}^1 = (\sigma^2)_{a,i}^1 = \infty$

for  $a \neq S$  and  $s = i$ .

3. Consider each  $a$  and all arcs of the form  $(a, b)$ .

For each  $j$  consider the quantity:

$$L = \mu_{b,i-s}^j + l(a, b) - \mu(a, b, s) + \sqrt{(\sigma^2)_{b,i-s}^j + \sigma^2(a, b, s)} \sqrt{\frac{1}{\beta} - 1}$$

and delete from the set  $i$  at node  $a$  all labels  $j'$  which satisfy:

$$\mu_{a,i}^{j'} \geq \mu_{b,i-s}^j + l(a, b) - \mu(a, b, s)$$

$$\mu_{a,i}^{j'} + \sigma_{a,i}^{j'} \sqrt{\frac{1}{\beta} - 1} \geq L$$

with strict inequality for at least one of these. If there is no  $j'$  satisfying:

$$\mu_{a,i}^{j'} \leq \mu_{b,i-s}^j + l(a, b) - \mu(a, b, s)$$

$$\mu_{a,i}^{j'} + \sigma_{a,i}^{j'} \sqrt{\frac{1}{\beta} - 1} \leq L$$

introduce the label  $(\mu, \sigma^2, t, k)_{a,i}^-$  where

$$\mu_{a,i}^- = \mu_{b,i-s}^j + l(a,b) - \mu(a,b,s)$$

$$(\sigma^2)_{a,i}^- = (\sigma^2)_{b,i-s}^- + \sigma^2(a,b,s)$$

$$t_{a,i}^- = b$$

$$k_{a,i}^- = s$$

If any changes result in the labels after examining an arc, re-rank the labels in order of increasing first component. This results in a change in the superscript  $j$  for some labels. In this set,  $j$  now takes on the values  $1, 2, \dots, r$  where  $r$  is the number of labels in set  $i$  at node  $a$ .

4. If  $s = 0$  go to 5. Otherwise, decrease  $s$  by 1 and go back to 3.

5. Repeat 3 until no changes in the labels of set  $i$  at node  $a$  result.

6. If  $i < n$ , increase  $i$  by 1 and go back to 2. If  $i = n$ , terminate, as  $\mu_{\bar{S},n}^{\bar{J}}$  and  $(\sigma^2)_{\bar{S},n}^{\bar{J}}$  are the mean and variance of the length of the desired path where  $\bar{J}$  equals the number of members of set  $n$  at  $\bar{S}$ .

The path itself may be found by the following procedure:

1. Set  $m = 1$ ,  $a_1 = \bar{S}$ , and  $i_1 = n$ .  
 2. Let  $(\mu, \sigma^2, t, k)_{\bar{S},n}^{\bar{J}_1}$  be the label of highest rank among set  $n$  at  $\bar{S}$ .

3. Set  $k_m = k_{a_m, i_m}^{j_m}$

4. Increase  $m$  by 1. Set  $i_m = i_{m-1} - k_{m-1}$  and

$a_m = t_{a_{m-1}, i_{m-1}}^{j_{m-1}}$ . If  $a_m = S$ , go to 6. Otherwise, find a

label in set  $i_m$  at node  $a_m$  which satisfies<sup>(1)</sup>

$$\mu_{a_m, i_m}^{j_m} = \mu_{a_{m-1}, i_{m-1}}^{j_{m-1}} - l(a_{m-1}, a_m) + \mu(a_{m-1}, a_m, k_{m-1})$$

$$(\sigma^2)_{a_m, i_m}^{j_m} = (\sigma^2)_{a_{m-1}, i_{m-1}}^{j_{m-1}} - \sigma^2(a_{m-1}, a_m, k_{m-1})$$

5. Go back to 3.

6. Terminate.  $S = a_m, \dots, a_1 = \bar{S}$  with arc  $(a_j, a_{j+1})$  having  $k_j$  improvements is the desired path. The arcs of the primal upon which breakdowns must occur, are those which intersect this path. Specifically,  $k_j$  breakdowns must occur on the primal arc which intersects arc  $(a_j, a_{j+1})$  in this path.

### 3. Justification of the Algorithm

The justification will consist in showing that the algorithm is finite and that at termination the label of highest rank in set  $n$  at  $\bar{S}$  has as its first two components the mean and variance of the length of the  $n$ -arc path to the sink of minimum effective length. Furthermore, the steps of the tracing procedure can be carried out and finds this path.

Lemma 1: Let  $S = a_0, a_1, \dots, a_m$  with  $k_j$  breakdowns occurring on arc  $(a_{j-1}, a_j)$  be an  $i$ -arc path to  $a_m$  such that all of its arcs are distinct. Let the mean and variance of the length of arc  $(a_{j-1}, a_j)$  be  $\mu_j$  and " $\sigma_j^2$ " respectively. Then the total

<sup>(1)</sup> Note that  $\mu(a, b, 0) = \sigma^2(a, b, 0) = 0$ .

length of this path is a random variable whose mean is equal to

$$\sum_{j=1}^m \mu_j \quad \text{and whose variance is equal to} \quad \sum_{j=1}^m \sigma_j^2.$$

Proof: The amounts by which the lengths of the different arcs can be reduced are independent. Hence the lengths of the arcs themselves are independent and their variances may be added to give the variance of the total length of the path. Of course, the means may be added regardless of whether or not the distributions are independent.

Corollary 2: Let  $S = a_0, \dots, a_m$  with  $k_j$  breakdowns occurring on arc  $(a_{j-1}, a_j)$  be a non-cyclic  $i$ -arc path to node  $a_m$ , and let the length of arc  $(a_{j-1}, a_j)$  have mean  $\mu_j$  and variance  $\sigma_j^2$ . Then the length of this path has mean

$$\sum_{j=1}^m \mu_j \quad \text{and variance} \quad \sum_{j=1}^m \sigma_j^2.$$

Proof: Since the path is non-cyclic, all its arcs are distinct and the corollary follows from lemma 1.

Definition: The pair  $(\mu, \sigma^2)$  is said to dominate the pair  $(\bar{\mu}, \bar{\sigma}^2)$  if and only if:

- i)  $\mu \leq \bar{\mu}$
- ii)  $\mu + \sigma \sqrt{\frac{1}{\beta} - 1} \leq \bar{\mu} + \bar{\sigma} \sqrt{\frac{1}{\beta} - 1}$

with strict inequality for at least one of these. If path 1 is an  $i$ -arc path to node  $a$  whose length has mean  $\mu_1$  and variance  $\sigma_1^2$ , and path 2 is an  $i$ -arc path to node  $a$  whose length has mean  $\mu_2$  and variance  $\sigma_2^2$  then path 1 is said to dominate path

2 if and only if the pair  $(\mu_1, \sigma_1^2)$  dominates  $(\mu_2, \sigma_2^2)$ . An  $i$ -arc path to node  $a$  that is not dominated by any other  $i$ -arc path to node  $a$  is said to be an undominated  $i$ -arc path. Furthermore, a label  $(\mu, \sigma^2, t, k)_{a,i}^j$  dominates the label  $(\mu, \sigma^2, t, k)_{a,i}^{\bar{j}}$  if and only if the pair  $(\mu_{a,i}^j, (\sigma^2)_{a,i}^j)$  dominates  $(\mu_{a,i}^{\bar{j}}, (\sigma^2)_{a,i}^{\bar{j}})$ .

Note that the dominance property is transitive, that a label can be introduced into a set only if no other label in that set dominates it, and finally, when a label is being considered for introduction into a set, the labels that are dropped from that set are precisely those which are dominated by the one being considered.

Lemma 3: A label cannot dominate another label in the set.

Proof: Suppose the theorem holds at one stage of the algorithm and that node  $a$  and arc  $(a, b)$  with  $s$  improvements are being considered with respect to the label  $(\mu, \sigma^2, t, k)_{b,i-s}^j$ .

Let:

$$\begin{aligned}\bar{\mu} &= \mu_{b,i-s}^j + l(a, b) - \mu(a, b, s) \\ \bar{\sigma}^2 &= (\sigma^2)_{b,i-s}^j + \sigma^2(a, b, s)\end{aligned}$$

All labels among set  $i$  at node  $a$  whose first two components are dominated by  $(\bar{\mu}, \bar{\sigma}^2)$  are dropped from this set and the condition still holds. Then the label  $(\mu, \sigma^2, t, k)_{a,i}^{\bar{j}} = (\bar{\mu}, \bar{\sigma}^2, b, s)$  is introduced into this set if and only if it is not dominated by any other label in this set and the property is still preserved. The only other way a change in labels can result is through the introduction of  $(0, 0, -, -)$  or  $(\infty, \infty, -, -)$  into an empty set. Initially, the conditions of the lemma hold with all sets empty and the lemma follows

from induction.

Lemma 4: A label can only be dropped from a set if another label which dominates it is introduced into that set.

Proof: Suppose a label  $(\mu, \sigma^2, t, k)_{a,i}^j$  is to be dropped from set  $i$  at node  $a$ . Then there is a quadruple  $(\bar{\mu}, \bar{\sigma}^2, \bar{t}, \bar{k})$  which will be introduced as a label into this set provided no other label would dominate it and which has the property that it dominates  $(\mu, \sigma^2, t, k)_{a,i}^j$ . Furthermore, no other label in this set can dominate it for if it did it would also dominate  $(\mu, \sigma^2, t, k)_{a,i}^j$ . Hence  $(\mu, \sigma^2, t, k)_{a,i}^j = (\bar{\mu}, \bar{\sigma}^2, \bar{t}, \bar{k})$  is introduced into set  $i$  at node  $a$  proving the theorem.

Corollary 5: If a label is dropped from a particular set, there will always be a label in that set which dominates it.

Proof: This follows immediately from lemma 4 and the transitivity of the dominance property.

Lemma 6: Let path 1,  $S = a_0, \dots, a_m = a$  with  $k_j$  improvements on arc  $(a_{j-1}, a_j)$  and path 2,  $S = \bar{a}_0, \dots, \bar{a}_m = a$  with  $\bar{k}_j$  improvements on arc  $(\bar{a}_{j-1}, \bar{a}_j)$  by non-cyclic  $i$ -arc paths to node  $a$ . Let path 1' and path 2' be  $(i+r)$ -arc paths to node  $b$  formed by adding arc  $(a, b)$  with  $r$  improvements to path 1 and path 2 respectively. If path 1 dominates path 2 then path 1' dominates path 2'.

Proof: Since paths 1 and 2 are non-cyclic, it follows that

the arcs of each path 1, 2, 1', and 2' are distinct. Letting the pairs  $(\mu(a), \sigma^2(a))$ ,  $(\bar{\mu}(a), \bar{\sigma}^2(a))$ ,  $(\mu(b), \sigma^2(b))$  and  $(\bar{\mu}(b), \bar{\sigma}^2(b))$  be the means and variances of the lengths of paths 1, 2, 1' and 2', it follows from lemma 1 that:

$$\mu(b) = \mu(a) + l(a, b) - \mu(a, b, r)$$

$$\bar{\mu}(b) = \bar{\mu}(a) + l(a, b) - \mu(a, b, r)$$

$$\sigma^2(b) = \sigma^2(a) + \sigma^2(a, b, r)$$

$$\bar{\sigma}^2(b) = \bar{\sigma}^2(a) + \sigma^2(a, b, r)$$

Therefore:

$$\mu(a) \leq \bar{\mu}(a)$$

$$\mu(a) + l(a, b) - \mu(a, b, r) \leq \bar{\mu}(a) + l(a, b) - \mu(a, b, r)$$

$$\mu(b) \leq \bar{\mu}(b)$$

with equality only if  $\mu(a) = \bar{\mu}(a)$  and the first condition is satisfied.

For the second condition one has:

Case 1:

$$\mu(a) + \sigma(a) \sqrt{\frac{1}{\beta} - 1} = \bar{\mu}(a) + \bar{\sigma}(a) \sqrt{\frac{1}{\beta} - 1}$$

$$\mu(a) < \bar{\mu}(a)$$

$$\sigma(a) > \bar{\sigma}(a)$$

$$\sigma^2(a) - \bar{\sigma}^2(a) = [\sigma^2(a) + \sigma^2(a, b, r)] - [\bar{\sigma}^2(a) + \sigma^2(a, b, r)]$$

$$\sigma^2(a) - \bar{\sigma}^2(a) = \sigma^2(b) - \bar{\sigma}^2(b)$$

$$\sigma(a) + \bar{\sigma}(a) \leq \sigma(b) + \bar{\sigma}(b)$$

$$\sigma(a) - \bar{\sigma}(a) \geq \sigma(b) - \bar{\sigma}(b)$$

$$\mu(a) + [\sigma(a) - \bar{\sigma}(a)] \sqrt{\frac{1}{\beta} - 1} = \bar{\mu}(a)$$

$$\mu(a) + [\sigma(b) - \bar{\sigma}(b)] \sqrt{\frac{1}{\beta} - 1} \leq \bar{\mu}(a)$$

$$\mu(b) + \sigma(b) \sqrt{\frac{1}{\beta} - 1} \leq \bar{\mu}(b) + \bar{\sigma}(b) \sqrt{\frac{1}{\beta} - 1}$$

Case 2:

$$\mu(a) + \sigma(a) \sqrt{\frac{1}{\beta} - 1} < \bar{\mu}(a) + \bar{\sigma}(a) \sqrt{\frac{1}{\beta} - 1}$$

Define  $\sigma$  such that

$$\begin{aligned} \mu(a) + \sigma \sqrt{\frac{1}{\beta} - 1} &= \bar{\mu}(a) + \bar{\sigma}(a) \sqrt{\frac{1}{\beta} - 1} \\ \mu(b) + \sqrt{\sigma^2 + \sigma^2(a, b, r)} \sqrt{\frac{1}{\beta} - 1} &\leq \bar{\mu}(b) + \bar{\sigma}(b) \sqrt{\frac{1}{\beta} - 1} \end{aligned}$$

Also  $\sigma(a) < \sigma$ . Thus:

$$\begin{aligned} \sigma(b) &< \sqrt{\sigma^2 + \sigma^2(a, b, r)} \\ \mu(b) + \sigma(b) \sqrt{\frac{1}{\beta} - 1} &< \bar{\mu}(b) + \bar{\sigma}(b) \sqrt{\frac{1}{\beta} - 1} \end{aligned}$$

**Lemma 7:** For each label  $(\mu, \sigma^2, t, k)_{a,i}^j$ , the quantities  $\mu_{a,i}^j$  and  $(\sigma^2)_{a,i}^j$  are either infinite or are equal to the mean and variance of an  $i$ -arc path to node  $a$ .

**Proof:** Suppose at one stage of the algorithm that for each label  $(\mu, \sigma^2, t, k)_{a,i}^j$  where  $\mu_{a,i}^j$  and  $(\sigma^2)_{a,i}^j$  are not infinite, there is an  $i$ -arc path  $S = a_0, \dots, a_m = a$  with  $k_j$  improvements on arc  $(a_{j-1}, a_j)$  whose arcs are distinct and whose length has mean  $\mu_{a,i}^j$  and variance  $(\sigma^2)_{a,i}^j$  and such that set  $[i - \sum_{j=n+1}^m k_j]$  at node  $a_n$  contains a label whose first two components equals or dominates the mean and variance of the length of the path  $S = a_0, \dots, a_n$  with  $k_j$  breakdowns on arc  $(a_{j-1}, a_j)$ . Suppose the label  $(\mu, \sigma^2, t, k)_{b,i+r}^j$  is introduced through the examination of arc  $(a, b)$  with  $r$  improvements. Then there is a label  $(\mu, \sigma^2, t, k)_{a,i}^j$  such that:

$$\begin{aligned} \mu_{b,i+r}^j &= \mu_{a,i}^j + l(a, b) - \mu(a, b, r) \\ (\sigma^2)_{b,i+r}^j &= (\sigma^2)_{a,i}^j + \sigma^2(a, b, r) \end{aligned}$$

Let  $S = a_0, \dots, a_m = a$  with  $k_j$  improvements on arc  $(a_{j-1}, a_j)$  be a path which satisfies the above conditions for  $(\mu, \sigma^2, t, k)_{a,i}^j$ . Consider the path  $S = a_0, \dots, a_{m+1} = b$  with  $k_j$  improvements on arc  $(a_{j-1}, a_j)$  .  $k_{m+1} = r$  . If each of its arcs are distinct, it satisfies the above conditions for  $(\mu, \sigma^2, t, k)_{b,i+r}^j$ . Suppose its arcs are not all distinct. Then there is an  $n < m$  such that  $a_n = a_m$  and  $a_{n+1} = a_{m+1}$ . The mean and variance of the length of above path  $S = a_0, \dots, a_{n+1}$  with  $k_j$  improvements on arc  $(a_{j-1}, a_j)$  for  $j \leq n$  and  $\sum_{j=n+1}^{m+1} k_j$  improvements on arc  $(a_n, a_{n+1})$  dominates the pair  $(\mu_{b,i+r}^j, (\sigma^2)_{b,i+r}^j)$ . Also the path  $S = a_0, \dots, a_n = a$  with  $k_j$  improvements on arc  $(a_{j-1}, a_j)$  dominates the path  $S = a_0, \dots, a_m$  with  $k_j$  improvements on arc  $(a_{j-1}, a_j)$ . Therefore, if  $\sum_{j=n+1}^m k_j = 0$  there is a label in set  $i$  at node  $a$  which dominates  $(\mu, \sigma^2, t, k)_{a,i}^j$  contradicting lemma 3. On the other hand, suppose  $\sum_{j=n+1}^m k_j \neq 0$ . Consider any label in set  $[i - \sum_{j=n+1}^m k_j]$  at node  $a$  whose first two components dominate the mean and variance of the length of  $S = a_0, \dots, a_n = a$  with  $k_j$  improvements on arc  $(a_{j-1}, a_j)$ . This label and arc  $(a, b)$  with  $\sum_{j=n+1}^{m+1} k_j > r$  improvements will be examined prior to the introduction of  $(\mu, \sigma^2, t, k)_{b,i+r}^j$  as a label. After this examination there must be a label among set  $(i+r)$  at node  $a_{m+1} = b$  which equals or dominates  $(\mu, \sigma^2, t, k)_{b,i+r}^j$ . But this prevents its introduction as a label.

Thus the arcs of  $S = a_0, \dots, a_{m+1} = b$  are distinct. The introduction of  $(0, 0, -, -)$  into any set at the source or  $(\infty, \infty, -, -)$  into any set preserves the above properties. Furthermore, the dropping of a label preserves these properties since it is immediately followed by the introduction of a label into its set which dominates it. Initially these properties hold with all sets of labels being empty. The lemma follows from induction.

Lemma 8: The labeling algorithm is finite.

Proof: Since the first two components of a label are either infinite or are equal to the mean and variance of a path whose arcs are distinct, it follows that this pair must be selected from a finite set. In addition the choices for the  $t_{a,i}^j$  and  $k_{a,i}^j$  are finite. Thus the labels themselves are selected from a finite set. Furthermore, no label may be introduced into the same set more than once for if it is once dropped there is another label in the set which dominates it and prevents its re-entry. Thus the algorithm is finite.

Lemma 9: At termination no labels are infinite at any node to which a path exists.

Proof: Consider set  $i$  at node  $a$ . Let  $S = a_0, \dots, a_m = a$  be any path to  $a$ . It follows that on or before the  $(m+1)$ <sup>st</sup> examination of the nodes for set  $i$  in step 5 a finite label will be introduced into the set  $i$  at node  $a$ .

Corollary 10: At termination  $\mu_{a,i}^j$  and  $(\sigma^2)_{a,i}^j$  correspond to the mean and variance of  $i$ -arc paths to node  $a$  for all  $i$  and all  $j$  provided a path to  $a$  exists.

Proof: This follows immediately from lemma 9.

Lemma 11: Let  $S = a_0, \dots, a_m = a$  with  $k_j$  improvements on arc  $(a_{j-1}, a_j)$  be an undominated  $i$ -arc path to node  $a$ . Then this path contains no cycles.

Proof: Suppose this path does contain a cycle. Let  $(\bar{a}_{j-1}, \bar{a}_j)$ ,  $j = 1, \dots, \bar{m}$  be the distinct arcs of this path. Furthermore, let arc  $(\bar{a}_{j-1}, \bar{a}_j)$  be used  $C_j$  times in this path and let its length have mean  $\mu_j$  and variance  $\sigma_j^2$ . Then the total length of this path has mean  $\sum_{j=1}^{\bar{m}} C_j \mu_j$  and variance  $\sum_{j=1}^{\bar{m}} C_j^2 \sigma_j^2$ ,  $C_j \geq 1$  all  $j$ . Delete all cycles from this path. The resulting path is an  $i$ -arc path to node  $a$  whose length has mean  $\sum_{j=1}^{\bar{m}} \bar{C}_j \mu_j$  and variance  $\sum_{j=1}^{\bar{m}} \bar{C}_j^2 \sigma_j^2$ ,  $\bar{C}_j \leq 1$  all  $j$  and  $\bar{C}_j = 0$  for some  $j$ . Thus the path  $S = a_0, \dots, a_m = a$  is not undominated.

Lemma 12: Let  $S = a_0, \dots, a_m = a$  with  $k_j$  improvements on arc  $(a_{j-1}, a_j)$  be an undominated  $i$ -arc path to node  $a$ . Then  $S = a_0, \dots, a_{m-1}$  with  $k_j$  improvements on arc  $(a_{j-1}, a_j)$  is an undominated  $(i-k_m)$ -arc path to node  $a_{m-1}$ .

Proof: Suppose the above path is not undominated. Let  $S = b_0, \dots, b_{\bar{m}} = a_{m-1}$  with  $\bar{k}_j$  improvements on arc  $(b_{j-1}, b_j)$  be an undominated  $(i-k_m)$ -arc path to node  $a_{m-1}$  that dominates

$S = a_0, \dots, a_{m-1}$  with  $k_j$  improvements on arc  $(a_{j-1}, a_j)$ .  
 Then  $S = b_0, \dots, b_m$  and  $S = a_0, \dots, a_{m-1}$  contain no cycles  
 and it follows from lemma 6 that  $S = b_0, \dots, b_m, a_m$  with  $k_j$   
 improvements on arc  $(b_{j-1}, b_j)$  and  $k_m$  improvements on arc  
 $(b_m, a_m)$  is an  $i$ -arc path to node  $a_m$  which dominates  
 $S = a_0, \dots, a_m$  with  $k_j$  improvements on arc  $(a_{j-1}, a_j)$  contra-  
 dicting the hypothesis.

Lemma 13: At termination the set of pairs  $(\mu_{a,i}^j, (\sigma^2)_{a,i}^j)$   
 is the set of means and variances of all undominated  $i$ -arc paths  
 to node  $a$  provided a path to  $a$  exists.

Proof: Suppose that after the algorithm terminates for the  
 sets  $r$  the lemma holds for all sets of pairs  $(\mu_{a,i}^j, (\sigma^2)_{a,i}^j)$ ,  
 $i \leq r$ . Let  $S = a_0, \dots, a_m = a$  with  $k_j$  improvements on arc  
 $(a_{j-1}, a_j)$  be an undominated  $(r+1)$ -arc path to node  $a$  and let  
 $(a_{p-1}, a_p)$  be the last arc in this path upon which improvements  
 occur. From repeated application of lemma 12 it follows that  
 $S = a_0, \dots, a_{p-1}$  with  $k_j$  improvements on arc  $(a_{j-1}, a_j)$  is an  
 undominated  $(r+1-k_p)$ -arc path to node  $a_{p-1}$ . Let the mean and  
 variance of the lengths of these two paths be the pairs  $(\mu_m, \sigma_m^2)$   
 and  $(\mu_{p-1}, \sigma_{p-1}^2)$  respectively. Then there exists a pair  
 $(\mu_{a_{p-1}, r+1-k_p}^j, (\sigma^2)_{a_{p-1}, r+1-k_p}^j) = (\mu_{p-1}, \sigma_{p-1}^2)$ . (i.e., there is a  
 label in set  $r+1-k_p$  at node  $a_{p-1}$  whose first two components  
 are  $\mu_{p-1}$  and  $\sigma_{p-1}^2$ .) Note that the path  $S = a_0, \dots, a_p$  with  
 $k_j$  improvements on arc  $(a_{j-1}, a_j)$  is an undominated  $(r+1)$ -arc

path to node  $a_p$ . Denote the mean and variance of the length of this path by  $\mu_p$  and  $\sigma_p^2$ . There can be no pair  $(\mu_{a_p, r+1}^j, (\sigma^2)_{a_p, r+1}^j)$  that dominates  $(\mu_p, \sigma_p^2)$  since such a pair must either be infinite or be equal to the mean and variance of an  $(r+1)$ -arc path to node  $a_p$ . Thus, after examining set  $(r+1-k_p)$  at node  $a_{p-1}$  and arc  $(a_{p-1}, a_p)$  with  $k_p$  improvements there must be a label  $(\mu, \sigma^2, t, k)_{a_p, r+1}^j$  such that  $\mu_{a_p, r+1}^j = \mu_p$  and  $(\sigma^2)_{a_p, r+1}^j = \sigma_p^2$ . Furthermore, this label can never be dropped. Repeated application of this argument yields the conclusion that eventually a label  $(\mu, \sigma^2, t, k)_{a, r+1}^j$  will be introduced where  $\mu_{a, r+1}^j = \mu_m$  and  $(\sigma^2)_{a, r+1}^j = \sigma_m^2$  and that it will never be dropped. Furthermore, any label in set  $r+1$  at node  $a$  whose first two components correspond to the mean and variance of a dominated  $(r+1)$ -arc path to node  $a$  will be dropped by the introduction of a label whose first two components correspond to the mean and variance of the length of a path that dominates the above mentioned one. Thus, when the algorithm terminates for the sets  $r+1$  the lemma holds for all sets of pairs  $(\mu_{a, i}^j, (\sigma^2)_{a, i}^j)$ ,  $i \leq r+1$ . Let  $S = b_0, \dots, b_{\bar{m}} = a$  be an undominated 0-arc path to node  $a$ . An argument similar to that above shows that after, at most,  $\bar{m}+1$  checks of the nodes there is a label  $(\mu, \sigma^2, t, k)_{a, 0}^j$  such that  $\mu_{a, 0}^j$  and  $(\sigma^2)_{a, 0}^j$  are the mean and variance of the length of this path, and finally, when the algorithm terminates for the sets 0 the lemma holds for all sets of pairs  $(\mu_{a, 0}^j, (\sigma^2)_{a, 0}^j)$ . The

lemma itself follows from induction.

Theorem 14: When the algorithm terminates, the label of highest rank among set  $n$  at  $\bar{S}$  has as its first two components the mean and variance of the  $n$ -arc path of minimum effective length to  $\bar{S}$ .

Proof: The  $n$ -arc path of minimum effective length is undominated and therefore there must exist a label  $(\mu, \sigma^2, t, k)_{\bar{S}, n}^j$  at termination such that  $\mu_{\bar{S}, n}^j$  and  $(\sigma^2)_{\bar{S}, n}^j$  correspond to the mean and variance of the length of this path. Furthermore, there can be no other label  $(\mu, \sigma^2, t, k)_{\bar{S}, n}^T$  of higher rank, for if so one has:

$$\mu_{\bar{S}, n}^j < \mu_{\bar{S}, n}^T$$

$$\mu_{\bar{S}, n}^j + \sigma_{\bar{S}, n}^j \sqrt{\frac{1}{\beta} - 1} \leq \mu_{\bar{S}, n}^T + \sigma_{\bar{S}, n}^T \sqrt{\frac{1}{\beta} - 1}$$

contradicting lemma 3.

Of course the  $n$ -arc path of minimum effective length to  $\bar{S}$  is the route of smallest effective length possible from source to sink due to at most  $n$  breakdowns. Thus the algorithm does find the mean and variance of the desired path. It now must be shown that the tracing procedure actually finds this path and the arcs upon which the  $n$  improvements must occur.

Lemma 15: Let  $(\mu, \sigma^2, t, k)_{a, i}^j$ ,  $a \neq \bar{S}$  be a label at termination. Then there is another label  $(\mu, \sigma^2, t, k)_{b, \bar{T}}^T$  at termination such that:

$$b = t_{a,i}^j$$

$$\bar{i} = i - k_{a,i}^j$$

$$\mu_{a,i}^j = \mu_{b,\bar{i}}^{\bar{j}} + l(a,b) - \mu(a,b,k_{a,i}^j)$$

$$(\sigma^2)_{a,i}^j = (\sigma^2)_{b,\bar{i}}^{\bar{j}} + \sigma^2(a,b,k_{a,i}^j)$$

Proof: Consider the label from which  $(\mu, \sigma^2, t, k)_{a,i}^j$  was obtained. This label must satisfy the above conditions. (i.e., must be  $(\mu, \sigma^2, t, k)_{b,\bar{i}}^{\bar{j}}$ ). Furthermore, this label cannot be dropped, for if it is another label  $(\mu, \sigma^2, t, k)_{b,\bar{i}}^{\bar{j}}$  is introduced which dominates it. Examining this label and arc  $(a,b)$  with  $k_{a,i}^j$  improvements produces a label which dominates and hence drops  $(\mu, \sigma^2, t, k)_{a,i}^j$ .

Lemma 16: The tracing procedure is finite.

Proof: The sequence of labels found is strictly decreasing lexicographically in  $(\mu_{a,i}^j, i)$  and hence none can be repeated. Thus the process is finite.

Lemma 17: Let the sequence of labels found by the tracing procedure be:

$$(\mu, \sigma^2, t, k)_{a_r, i_r}^{j_r}, \quad r = 1, \dots, m$$

Then:

$$\mu_{S,n}^{j_1} = \mu_{a_1, i_1}^{j_1} = \sum_{r=1}^{m-1} [l(a_r, a_{r+1}) - \mu(a_r, a_{r+1}, k_r)]$$

$$(\sigma^2)_{S,n}^{j_1} = (\sigma^2)_{a_1, i_1}^{j_1} = \sum_{r=1}^{m-1} \sigma^2(a_r, a_{r+1}, k_r)$$

where  $k_r = k_{a_r, i_r}^{j_r}$ .

Proof: It follows from the rules of the tracing procedure that:

$$\begin{aligned}
 (i) \quad & \mu_{a_r, i_r}^{j_r} - \mu_{a_{r+1}, i_{r+1}}^{j_{r+1}} = i(a_r, a_{r+1}) - \mu(a_r, a_{r+1}, k_r), \quad r = 1, \dots, m-1 \\
 & \mu_{S, i_m}^{j_m} = \mu_{a_m, i_m}^{j_m} = 0 \\
 (ii) \quad & (\sigma^2)_{a_r, i_r}^{j_r} - (\sigma^2)_{a_{r+1}, i_{r+1}}^{j_{r+1}} = \sigma^2(a_r, a_{r+1}, k_r), \quad r = 1, \dots, m-1 \\
 & (\sigma^2)_{S, i_m}^{j_m} = (\sigma^2)_{a_m, i_m}^{j_m} = 0.
 \end{aligned}$$

Summing the equations (i) give the first relationship and summing the equations (ii) gives the second.

Theorem 18: Let the sequence of labels found by the tracing procedure be as in lemma 17. Then  $S = a_m, \dots, a_1 = \bar{S}$  with  $k_r = k_{a_r, i_r}^{j_r}$  improvements on arc  $(a_r, a_{r+1})$  is an  $n$ -arc path from source to sink whose length has mean  $\mu_{S, n}^{j_1}$  and variance  $(\sigma^2)_{S, n}^{j_1}$ .

Proof: The quantities  $\mu_{S, n}^{j_1}$  and  $(\sigma^2)_{S, n}^{j_1}$  equal the sums of the means and variances respectively of the arcs in this path. All the arcs in this path are distinct, for if not there would be a cycle, and deleting all cycles would yield an  $n$ -arc path to  $\bar{S}$  such that its length has mean strictly less than  $\mu_{S, n}^{j_1}$  and variance not exceeding  $(\sigma^2)_{S, n}^{j_1}$  contradicting theorem 14. Thus the theorem follows from lemma 1.

This completes the justification of the procedure for finding

the desired route itself. The primal arcs upon which breakdowns must occur are those which intersect this route. In particular  $k_j$  breakdowns must occur upon that arc of the primal which intersects arc  $(a_j, a_{j+1})$ .

#### 4. Normal Case

Suppose it is known that the decrease in the capacities of the arcs due to breakdowns are independently and normally (or rather approximately normally) distributed with known mean and variance. It follows that for any breakdown pattern all routes through the dual are normally distributed.

Let the length of a particular path be a random variable  $L$  whose distribution is normal with mean  $\mu$  and variance  $\sigma^2$

Then:

$$P\left\{\frac{L - \mu}{\sigma} \geq \Phi(1 - \beta)\right\} = \beta$$

where  $\Phi(1 - \beta)$  satisfies

$$P\{Y \leq \Phi(1 - \beta)\} = 1 - \beta$$

where  $Y$  is standard normal. Thus:

$$P\{L \geq \mu + \sigma \Phi(1 - \beta)\} = \beta$$

Hence  $\mu + \sigma \Phi(1 - \beta)$  plays the role of the effective length of this path. Suppose it is desired to find the  $n$ -arc path to the sink of minimum effective length. Then the algorithm just presented in this chapter may be used to solve this problem by replacing

$$\sqrt{\frac{1}{\beta}} - 1 \text{ by } \Phi(1 - \beta).$$

## 5. Validity of the Bound

Under the breakdown pattern found by the algorithm the following relationship exists:

$$P\{V[A, B] \geq \bar{F}\} \leq \beta$$

where  $[A, B]$  is the cut set which intersects the  $n$ -arc path of minimum effective length through the dual and  $\bar{F}$  is the effective length of this path. However, the stochastic nature of this network indicates that  $[A, B]$  need not be the minimum cut with probability one and thus one is led to believe that there might exist an  $\bar{F}' < \bar{F}$  such that:

$$P\{\text{max flow} \geq \bar{F}'\} \leq \beta$$

In this section, a stochastic program is formulated which applies Tchebycheff's extended lemma to the entire network. It is then shown, surprisingly enough, that such an  $\bar{F}'$  does not exist for this program.

This linear programming formulation of the network is:

Find  $x_{ij} \geq 0$ , max  $F$  such that:

$$x_{ij} \leq b_{ij}$$

$$(1) \quad \begin{aligned} \sum_j x_{jS} - \sum_j x_{Sj} + F &= 0 \\ \sum_j x_{ji} - \sum_j x_{ij} &= 0, \quad i \neq S, \quad \bar{S} \\ \sum_j x_{jS} - \sum_j x_{\bar{S}j} - F &= 0 \end{aligned}$$

The dual to this program is:

Find  $u_{ij} \geq 0$ ,  $v_i$ ,  $\min Z$  such that:

$$u_{ij} - v_i + v_j \geq 0$$

$$(2) \quad v_S - v_{\bar{S}} \geq 1$$

$$\sum_{i,j} b_{ij} u_{ij} = Z$$

where  $b_{ij}$  is a random variable. Of course, by the duality theorem of linear programming,  $\min Z = \max F$  for any fixed values of the  $b_{ij}$ .<sup>(1)</sup> Let  $\mu_{ij}$  and  $\sigma_{ij}^2$  be the mean and variance of the amount by which the capacity of arc  $(i,j)$  is reduced by the given breakdown pattern and let  $C_{ij}$  be its original capacity. Then:

$$E(b_{ij}) = C_{ij} - \mu_{ij}$$

$$\text{var}(b_{ij}) = \sigma_{ij}^2$$

Applying Tchebycheff's extended lemma, the stochastic program (2) can be reduced to the nonlinear program (3).<sup>(2)</sup>

Find  $u_{ij} \geq 0$ ,  $v_i$ ,  $\min Z$  such that:

$$-u_{ij} + v_i - v_j \leq 0, \quad (i,j) \neq (S, \bar{S})$$

$$(3) \quad -v_S + v_{\bar{S}} + 1 \leq 0$$

$$\sum_{i,j} (C_{ij} - \mu_{ij}) u_{ij} + \sqrt{\frac{1}{\beta} - 1} \sqrt{\sum_{i,j} \sigma_{ij}^2 u_{ij}} = Z$$

<sup>(1)</sup>Reference 4, pp. 128-134.

<sup>(2)</sup>Reference 11, pp. 2-6.

**Theorem 1:** In (3)  $\min \bar{Z} \leq \bar{F}$

**Proof:** Let

$$u_{ij} = \begin{cases} 1 & \text{if } i \in A, j \in B \\ 0 & \text{otherwise} \end{cases}$$

$$v_i = \begin{cases} 1 & \text{if } i \in A \\ 0 & \text{if } i \in B \end{cases}$$

This satisfies the constraints of (3) and gives  $\bar{Z} = \bar{F}$ .

It now remains to determine whether or not  $\min \bar{Z} = \bar{F}$ . If  $\min \bar{Z} = \bar{F}$ , then the feasible solution to (3) given in the proof of theorem 1 is also an optimal solution. This happens if and only if there exists  $x_{ij} \geq 0$ ,  $F \geq 0$  such that:<sup>(1)</sup>

$$(4) \quad G(\hat{u}, \hat{v}) = \min G(u, v) =$$

$$\min_{u \geq 0} \bar{Z} + \sum_{(i,j) \neq (S, \bar{S})} x_{ij}(-u_{ij} + v_i - v_j) + F(-v_S + v_{\bar{S}} + 1)$$

where  $u = [u_{ij}]$ ,  $v = (v_i)$  and  $(\hat{u}, \hat{v})$  is the feasible solution to (3) given in the proof of theorem 1.

This is satisfied if and only if:

$$(5) \quad \left[ \sum_{i,j} \frac{\partial G(u, v)}{\partial u_i} (u_i - \hat{u}_i) + \sum_i \frac{\partial G(u, v)}{\partial v_i} (v_i - \hat{v}_i) \right] \geq 0$$

for all  $u \geq 0$  at  $(u, v) = (\hat{u}, \hat{v})$ . This happens if and only if:

$$\frac{\partial G(u, v)}{\partial u_i} \bigg|_{(u, v) = (\hat{u}, \hat{v})} \geq 0 \quad \text{for } u_i = 0$$

$$(6) \quad \frac{\partial G(u, v)}{\partial u_i} \bigg|_{(u, v) = (\hat{u}, \hat{v})} = 0 \quad \text{for } u_i > 0$$

$$\frac{\partial G(u, v)}{\partial v_i} \bigg|_{(u, v) = (\hat{u}, \hat{v})} = 0 \quad \text{all } i.$$

<sup>(1)</sup>Reference 4, pp. 471-472.

Performing the differentiation indicated in (6) one obtains:

$$x_{ij}, F \geq 0$$

$$x_{ij} = (C_{ij} - \mu_{ij}) + \frac{\sigma_{ij}^2}{\sqrt{\sum_{\substack{i \in A \\ j \in B}} \sigma_{ij}^2}} \sqrt{\frac{1}{\beta} - 1}, \quad i \in A, j \in B$$

$$(7) \quad x_{ij} \leq (C_{ij} - \mu_{ij}) + \frac{\sigma_{ij}^2}{\sqrt{\sum_{\substack{i \in A \\ j \in B}} \sigma_{ij}^2}} \sqrt{\frac{1}{\beta} - 1} \text{ otherwise}$$

$$\sum_j x_{Sj} - \sum_j x_{jS} - F = 0$$

$$\sum_j x_{ij} - \sum_j x_{ji} = 0, \quad i \neq S, \bar{S}$$

$$\sum_j x_{\bar{S},j} - \sum_j x_{j\bar{S}} + F = 0$$

It will be shown in the proof of the following lemma that the relationships (7) can be satisfied if and only if a certain network which is related to the primal network has  $[A, B]$  as a minimum cut. Later on it will be shown that  $[A, B]$  is a minimum cut of this network and therefore (7) can be satisfied and  $\min \bar{Z} = \bar{F}$ .

Lemma 2: Consider the network formed by taking the primal network and changing the capacity of arc  $(i, j)$  to

$$\left[ (C_{ij} - \mu_{ij}) + \frac{\sigma_{ij}^2}{\sqrt{\sum_{\substack{i \in A \\ j \in B}} \sigma_{ij}^2}} \right] . \quad (\text{Note that } \mu_{ij} = \sigma_{ij}^2 = 0 \text{ if no break-}$$

downs occur on arc  $(i, j)$ ). Then  $\min \bar{Z} = \bar{F}$  in (3) if and only if  $[A, B]$  is a minimum cut of this network.

Proof: If  $\min \bar{Z} = \bar{F}$  in (3) then the  $x_{ij}$ ,  $\bar{F}$  which satisfy (7) also satisfy the constraints of the network and produce a flow of  $\bar{F} = V[A, B]$ . Thus  $[A, B]$  is a minimum cut. On the other hand, if  $[A, B]$  is a minimum cut, then, letting  $x_{ij}$  be the flow along arc  $(i, j)$  the  $x_{ij}$  of any maximal flow pattern and  $\bar{F}$  satisfy (7).

Corollary 3: If  $\min \bar{Z} \neq \bar{F}$  in (3) then the network in theorem 2 has a cut  $[C, D]$  such that  $V[C, D] < \bar{F}$ .

Proof: This follows immediately from lemma 2.

Definition: Let the effective capacity of a set of arcs,  $A$  be  $\mu + \sigma \sqrt{\frac{1}{\beta}} - 1$  where  $\mu$  and  $\sigma^2$  are the mean and variance of the sum of the capacities of the arcs in  $A$  respectively.

Note that all properties of effective lengths of arcs in the dual hold for the effective capacities of the corresponding arcs in the primal. Hence the analogous versions of the theorems in Chapter 4, section 3, on effective length will be assumed to hold for effective capacity.

Lemma 4: The number of breakdowns occurring on cut  $[A, B]$  is  $n$ .

Proof: Suppose only  $n - k$ ,  $k > 0$ , breakdowns occur on cut  $[A, B]$ . Increase the number of breakdowns on any arc of  $[A, B]$  by  $k$ . This results in a decrease in the effective capacity of  $[A, B]$  and hence in the effective length of the corresponding route through the dual network, contradicting theorems 14 and 18 of Chapter 4, section 3.

Lemma 5: If  $\min \bar{Z} < \bar{F}$  in (3) then under the breakdown pattern found by the algorithm there is a cut  $[C, D]$  such that the effective capacity of the arcs of  $\{[C, D] - [A, B] \cap [C, D]\}$  is strictly less than the effective capacity of the arcs of  $\{[A, B] - [A, B] \cap [C, D]\}$ .

Proof: From corollary 3 there exists a cut  $[C, D]$  such that:

$$C_1 - \mu_1 + \frac{\sigma_1^2}{\sigma_2} \sqrt{\frac{1}{\beta} - 1} < C_2 - \mu_2 + \sigma_2 \sqrt{\frac{1}{\beta} - 1}$$

where  $C_1$  is the sum of the zero-breakdown capacities of the arcs in  $[C, D]$ ,  $\mu_1$  and  $\sigma_1^2$  are the mean and variance of the reduction in this sum due to the breakdown pattern and  $C_2$ ,  $\mu_2$ , and  $\sigma_2^2$  are these same quantities for  $[A, B]$ . Let  $C$ ,  $\mu$ , and  $\sigma^2$  be these quantities for  $[A, B] \cap [C, D]$ . From lemma 4,  $\mu = \mu_1$  and  $\sigma^2 = \sigma_1^2$ . Thus the effective capacity of  $\{[C, D] - [A, B] \cap [C, D]\}$  is  $C_1 - C$  and that of  $\{[A, B] - [A, B] \cap [C, D]\}$  is

$$[(C_2 - C) - (\mu_2 - \mu_1) + \sqrt{\sigma_2^2 - \sigma_1^2} \sqrt{\frac{1}{\beta} - 1}]$$

and one has:

$$\begin{aligned}
(C_1 - C) &< (C_2 - C) - (\mu_2 - \mu_1) + \left[ \sigma_2^2 - \frac{\sigma_1^2}{\sigma_2} \right] \sqrt{\frac{1}{\beta} - 1} \\
&\leq (C_2 - C) - (\mu_2 - \mu_1) + \sqrt{\sigma_2^2 - \sigma_1^2} \sqrt{\frac{1}{\beta} - 1}
\end{aligned}$$

proving the theorem.

**Lemma 6:** The mean and variance of the sum of the capacities of the arcs of  $\{[C, D] - [A, B] \cap [C, D]\}$  dominates the mean and variance of the sum of the capacities of the arcs of  $\{[A, B] - [A, B] \cap [C, D]\}$ .

**Proof:** Using the terminology of theorem 5 one has:

$$\begin{aligned}
(C_1 - C) &< (C_2 - C) - (\mu_2 - \mu_1) + \sqrt{\sigma_2^2 - \sigma_1^2} \sqrt{\frac{1}{\beta} - 1} \\
\text{and since } (\mu_2 - \mu_1) &= \sqrt{\sigma_2^2 - \sigma_1^2} \sqrt{\frac{1}{\beta} - 1} > 0 \text{ it follows that} \\
(C_1 - C) &< (C_2 - C) \text{ proving the theorem.}
\end{aligned}$$

**Theorem 7:** In the convex program (3)  $\min \bar{Z} = \bar{F}$ .

**Proof:** Assume  $\min \bar{Z} < \bar{F}$ . Then there is a cut  $[C, D]$  such that the mean and variance of the sum of the capacities of  $\{[C, D] - [A, B] \cap [C, D]\}$  dominates these same two quantities for  $\{[A, B] - [A, B] \cap [C, D]\}$ . But this means that the mean and variance of  $V[C, D]$  dominates that of  $V[A, B]$ , the proof of this being identical to that of lemma 6, chapter 4, section 3. This contradicts the fact that the algorithm finds the shortest  $n$ -arc path from source to sink.

Thus the solution to the stochastic program does not improve the bound found by the algorithm. This is a strong argument in favor of the validity of evaluating a breakdown pattern solely on the minimum effective capacity of all cut sets.

## 6. Practicality of the Algorithm

Since this algorithm is in many ways similar to the minimum route algorithm<sup>(1)</sup> which is highly efficient, it would seem that this algorithm would be practical, provided the number of labels in each set remained small. While examples have not been solved to establish whether or not the number of labels in each set is likely to remain small there are indications in favor of this happening.

The first of these indications is the assumption that:

$$\mu(a, b, i) > \mu(\bar{a}, \bar{b}, \bar{i})$$

implies

$$\mu(a, b, i) - \sigma(a, b, i) \sqrt{\frac{1}{\beta} - 1} > \mu(\bar{a}, \bar{b}, \bar{i}) - \sigma(\bar{a}, \bar{b}, \bar{i}) \sqrt{\frac{1}{\beta} - 1}.$$

Thus, the difference between the zero-breakdown length of a path and its effective length tends to increase as the mean decrease in length due to breakdowns increases.

The second of these indications is even more convincing. Let  $A$  and  $B$  be two paths to node  $a$  such that  $A$  has smaller mean length than  $B$  but does not dominate it. Then if arcs with breakdowns are added to the two paths, the new path formed from  $A$  tends to dominate that formed from  $B$ . This is summarized in theorem 1.

Theorem 1: Consider the following quantities:

$$\bar{\mu}_1 = \mu_1 + \mu$$

$$\bar{\mu}_2 = \mu_2 + \mu$$

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<sup>(1)</sup>Reference 7, pp.130-134.

$$\begin{aligned}
L_1 &= \mu_1 + |\sigma_1| \sqrt{\frac{1}{\beta} - 1} \\
L_2 &= \mu_2 + |\sigma_2| \sqrt{\frac{1}{\beta} - 1} \\
\bar{L}_1 &= \mu_1 + \mu + \sqrt{\sigma_1^2 + \sigma^2} \sqrt{\frac{1}{\beta} - 1} \\
\bar{L}_2 &= \mu_2 + \mu + \sqrt{\sigma_2^2 + \sigma^2} \sqrt{\frac{1}{\beta} - 1}
\end{aligned}$$

Then if  $\sigma_1^2 > \sigma_2^2$ ,  $\bar{\mu}_1 - \bar{\mu}_2 = \mu_1 - \mu_2$  and  $\bar{L}_1 - \bar{L}_2 < L_1 - L_2$  with equality if and only if  $\sigma^2 = 0$ .

Proof:

$\bar{\mu}_1 - \bar{\mu}_2 = \mu_1 - \mu_2$  follows trivially

$$\begin{aligned}
&(\bar{L}_1 - \bar{L}_2) - (L_1 - L_2) \\
&\left[ \sqrt{\sigma_1^2 + \sigma^2} - \sqrt{\sigma_2^2 + \sigma^2} - |\sigma_1| + |\sigma_2| \right] \sqrt{\frac{1}{\beta} - 1} \\
&|\sigma_1| > |\sigma_2| \\
&\sigma_1^2 + \sigma^2 - \sigma_2^2 = \sigma_2^2 + \sigma^2 - \sigma_2^2 \\
&\sqrt{\sigma_1^2 + \sigma^2} + |\sigma_1| > \sqrt{\sigma_2^2 + \sigma^2} + |\sigma_2| \\
&\sqrt{\sigma_1^2 + \sigma^2} - |\sigma_1| < \sqrt{\sigma_2^2 + \sigma^2} - |\sigma_2| \quad \text{if } \sigma^2 \neq 0 \\
&\sqrt{\sigma_1^2 + \sigma^2} - \sqrt{\sigma_2^2 + \sigma^2} - |\sigma_1| + |\sigma_2| < 0 \\
&(\bar{L}_1 - \bar{L}_2) - (L_1 - L_2) < 0
\end{aligned}$$

Note that equality holds in the above if  $\sigma^2 = 0$ .

The significance in this is that it tends to keep the number of labels in a set from increasing at a rapid rate as one gets further from the source.

## CHAPTER V

### Other Approaches

Two valid criticisms of the algorithms presented are that it would be more desirable to have an algorithm that worked for non-planar networks as well as planar networks and which worked directly with the primal network. These problems have been investigated and the purpose of this section is to discuss some of the difficulties encountered.

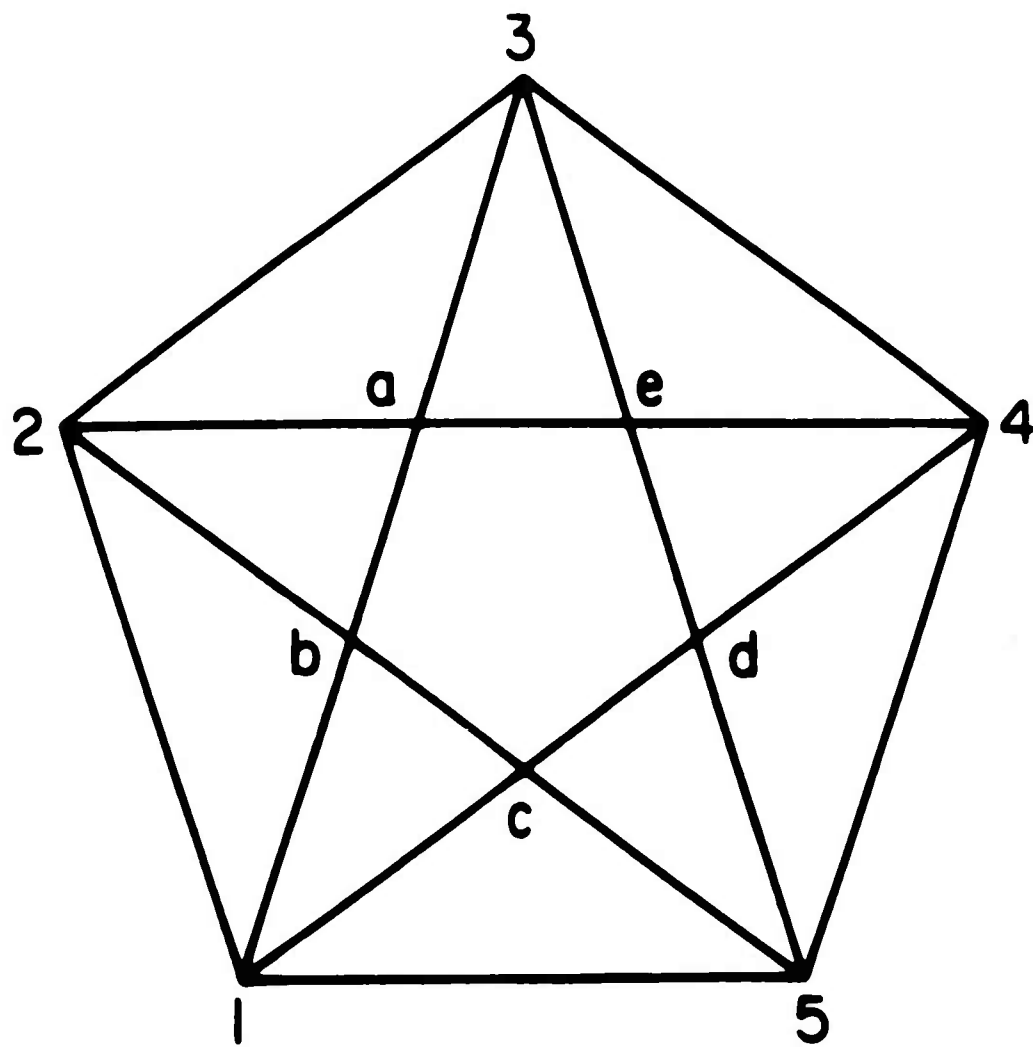
A labeling algorithm which works directly with the primal was developed by the author to find the arc which, when removed from the network, would reduce the maximum flow the most.<sup>(1)</sup> Among other things, this algorithm required a list of all the arcs. If this were to be extended to the case where  $n$  arcs are to be removed (or their capacities reduced), a list of all  $n$ -tuples of arcs would be required. Such a list would be too long to be practical.

An undesirable feature of this problem is that the solution for  $(n - 1)$  breakdowns does not supply useful information for the solution for  $n$  breakdowns. In fact, examples have been constructed where reducing the capacities of a particular set of  $n$  arcs reduces the maximum flow to zero while reducing the capacities of any  $(n - 1)$  arcs in this set causes no reduction in the maximum flow. While this produces no difficulties when working with routes through the dual, it appears to be an insurmountable barrier when working with cuts of the primal.

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<sup>(1)</sup>Reference 15.

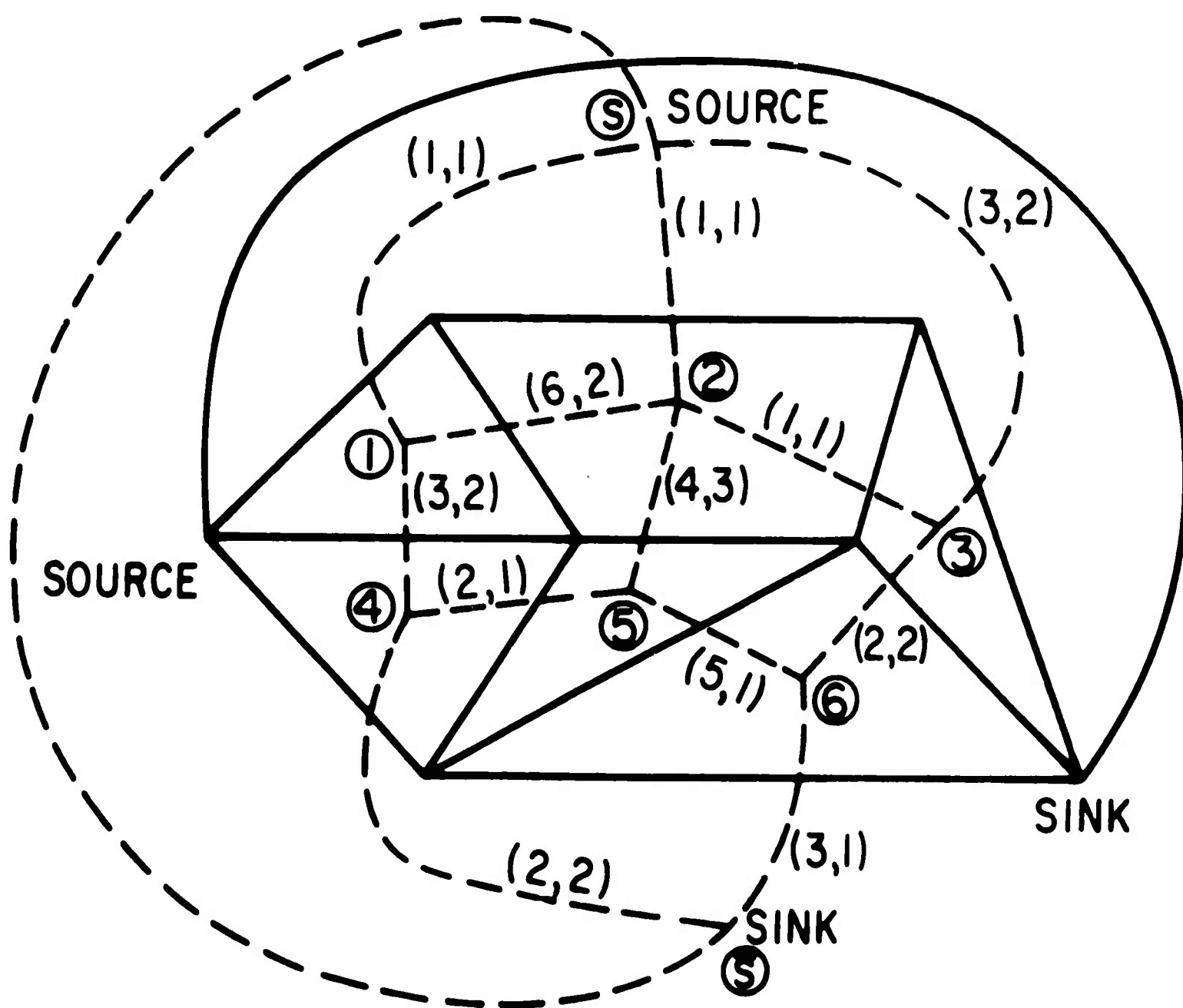
Assuming this problem cannot be solved directly using the primal network, one may be tempted to try to extend the notion of a dual network to non-planar networks. A natural way to do this is to draw the network on a sphere and place an artificial node wherever two arcs that are not joined by a node intersect. This essentially replaces each arc by one or more sub-arcs in such a way as to create a new network which is planar. Each sub-arc is assigned a capacity equal to that of the original arc it is a part of. A cut set  $[A, B]$  of the original non-planar network will be represented in the planar network if and only if there is a cut set consisting of exactly one sub-arc of each arc in  $[A, B]$ . However, some cut sets of the non-planar network may not be represented in the new planar one. An example of this is the network of figure 2. The actual nodes of the network are 1, 2, 3, 4, and 5 and the artificial ones are a, b, c, d, and e. Consider the cut  $[A, B]$  where  $A = \{1, 2, 4\}$  and  $B = \{3, 5\}$ . Suppose  $[A, B]$  is represented by a cut  $[A', B']$  in the modified network. It follows that  $1, 2, 4 \in A'$  and  $3, 5 \in B'$ . Since no sub-arc of (1, 4) can be in  $[A', B']$  it follows that  $c, d \in A'$  and since no sub-arc of (3, 5) is in  $[A', B']$  it follows that  $d, e \in B'$ . But this is impossible since  $A' \cap B' = \emptyset$  and therefore  $[A, B]$  is not represented.



**FIG. 2    A NON-PLANAR NETWORK WITH CUT SETS  
THAT ARE NOT REPRESENTED IN THE MODIFIED  
NETWORK. NUMBERS REPRESENT THE ACTUAL  
NODES AND LETTERS THE ARTIFICIAL NODES.**

### **Appendix I: Example for Deterministic Case**

An example was solved using the deterministic algorithm of Chapter III. The network itself is shown in figure 3 and the results of the calculations in table 1.



- (a,b) — ARC OF DUAL WITH LENGTH  $a$  AND REDUCTION OF LENGTH  $b$  IF BREAK-DOWN OCCURS
- ③ — ARC OF PRIMAL NODE  $c$  OF DUAL

FIG. 3 NETWORK FOR EXAMPLE 1

**Table 1: Results of Algorithm for Example 1**

**I. For  $(D, t, k)_{a, 0}$**

	<u>Initial</u>		<u>Iteration 1</u>		<u>Iteration 2</u>
a	$(D, t, k)_{a, 0}$	a	$(D, t, k)_{a, 0}$	a	$(D, t, k)_{a, 0}$
S	$(0, -, -)$	S	$(0, -, -)$	S	$(0, -, -)$
1	$(\infty, -, -)$	1	$(1, S, 0)$	1	$(1, S, 0)$
2	$(\infty, -, -)$	2	$(1, S, 0)$	2	$(1, S, 0)$
3	$(\infty, -, -)$	3	$(2, 2, 0)$	3	$(2, 2, 0)$
4	$(\infty, -, -)$	4	$(4, 1, 0)$	4	$(4, 1, 0)$
5	$(\infty, -, -)$	5	$(5, 2, 0)$	5	$(5, 2, 0)$
6	$(\infty, -, -)$	6	$(4, 3, 0)$	6	$(4, 3, 0)$
$\bar{S}$	$(\infty, -, -)$	$\bar{S}$	$(6, 4, 0)$	$\bar{S}$	$(6, 4, 0)$

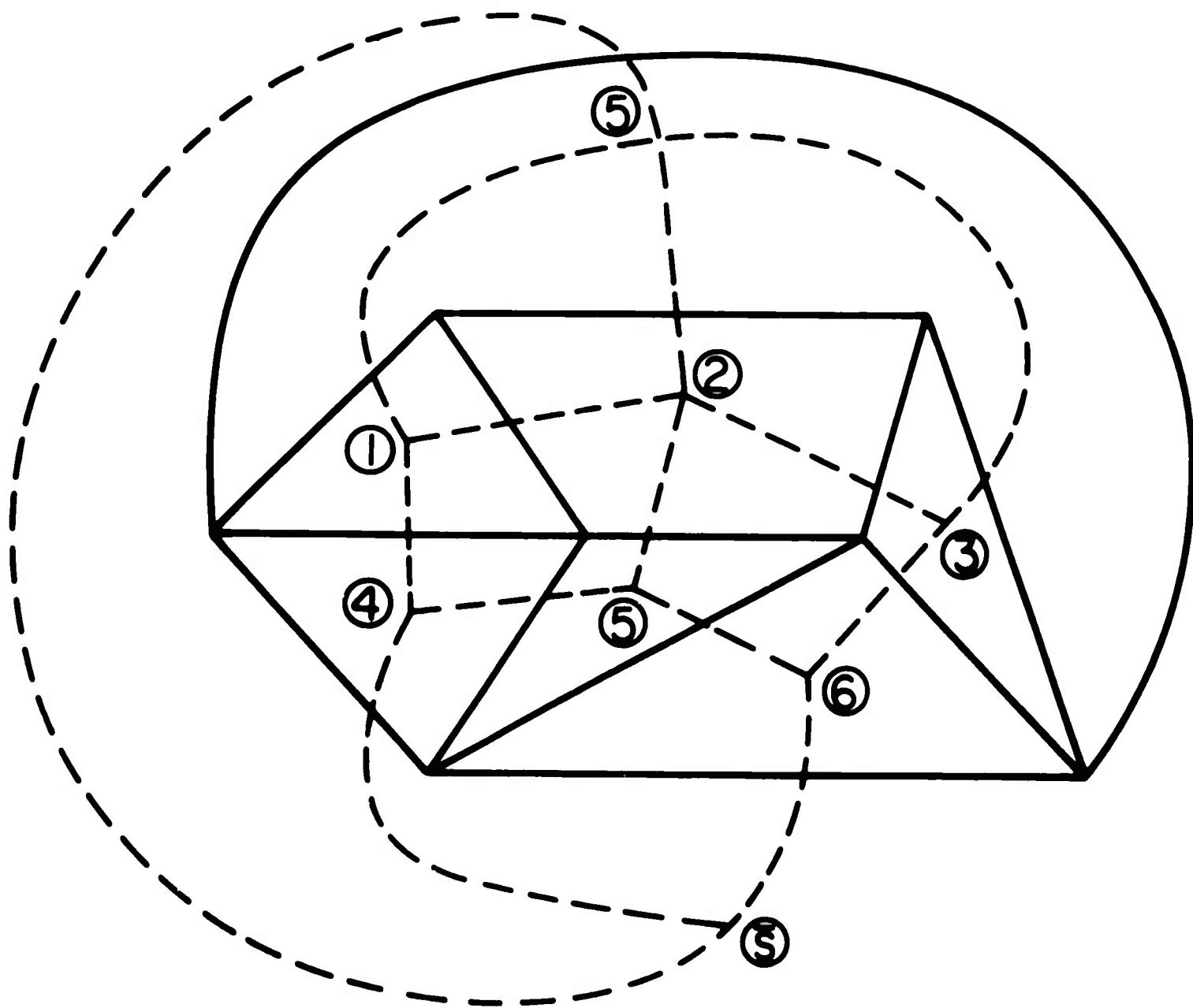
**II. For  $(D, t, k)_{a, 1}$**

	<u>Initial</u>		<u>Iteration 1</u>		<u>Iteration 2</u>
a	$(D, t, k)_{a, 1}$	a	$(D, t, k)_{a, 1}$	a	$(D, t, k)_{a, 1}$
S	$(0, -, -)$	S	$(0, -, -)$	S	$(0, -, -)$
1	$(\infty, -, -)$	1	$(0, S, 1)$	1	$(0, S, 1)$
2	$(\infty, -, -)$	2	$(0, S, 1)$	2	$(0, S, 1)$
3	$(\infty, -, -)$	3	$(1, S, 1)$	3	$(1, S, 1)$
4	$(\infty, -, -)$	4	$(2, 1, 1)$	4	$(2, 1, 1)$
5	$(\infty, -, -)$	5	$(2, 2, 1)$	5	$(2, 2, 1)$
6	$(\infty, -, -)$	6	$(2, 3, 1)$	6	$(2, 3, 1)$
$\bar{S}$	$(\infty, -, -)$	$\bar{S}$	$(4, 4, 0)$	$\bar{S}$	$(4, 4, 0)$

Path is S , 1, 4 ,  $\bar{S}$  with breakdown on (1, 4) .

## Appendix II: Example for Stochastic Case

Another example was solved using the stochastic algorithm of Chapter IV. The network is shown in figure 4 and the properties of the arcs in table 2. Each arc is subject to at most 1 breakdown. The decrease in capacity of an arc due to a breakdown is  $l(p)$  with probability  $p$  and  $l(1 - p)$  with probability  $(1 - p)$ . These parameters of course are different for different arcs. The problem was solved for  $\beta = \frac{1}{5}$  and the results summarized in tables 3 and 4. The path found is  $S, 1, 4, \bar{5}$  with a breakdown on  $(4, \bar{5})$ . The effective length of this path is  $\frac{26}{6} + 2\sqrt{\frac{1}{18}} \approx 4.9$ . Since a reduction in the length of  $(4, \bar{5})$  is at least  $\frac{3}{2}$  and the probability that this reduction is not greater than  $\frac{3}{2}$  is  $\frac{1}{3} > \frac{1}{5}$ , it follows that the true value of  $F$  is 4.5. Note that if  $\beta = \frac{1}{3}$  the effective length of this path and the true value of  $F$  for this path (but not necessarily for the entire network) are both 4.5.



——— ARC OF DUAL  
 - - - - - ARC OF PRIMAL  
 ① NODE a OF DUAL

FIG. 4 NETWORK FOR EXAMPLE 2

**Table 2: Data for Network of Figure 4**

<u>Arc</u>	<u>p</u>	<u>l(p)</u>	<u>l(1-p)</u>	<u>l(a, b)</u>	<u><math>\mu(a, b, l)</math></u>	<u><math>\sigma^2(a, b, l)</math></u>
(S, 1)	1/2	1	3/4	1	7/8	1/64
(S, 2)	1/4	5/8	3/4	1	23/32	13/4096
<b>(S, 3)</b>	1/3	4/3	11/6	3	5/3	1/18
(1, 2)	1/4	5	4	6	17/4	3/16
(1, 4)	1/5	1	7/4	3	8/5	9/100
(2, 3)	1/2	1/2	3/4	1	5/8	1/64
(2, 5)	1/5	2	3	4	14/5	4/25
(3, 6)	1/6	3/2	1	2	13/12	5/144
(4, 5)	1/3	1	3/2	2	4/3	1/18
(4, $\bar{5}$ )	1/3	3/2	2	2	11/6	1/18
(5, 6)	1/4	4	3	5	13/4	3/16
(6, $\bar{5}$ )	1/2	7/4	9/4	3	2	1/16

$$\beta = 1/5$$

**Table 3: Results of Algorithm for Example 2 - i = 0**

<u>Initial</u>		<u>Iteration 1</u>	
a	$(\mu, \sigma^2, t, k)_{a,0}^j$	a	$(\mu, \sigma^2, t, k)_{a,0}^j$
S	(0, 0, -, -)	S	(0, 0, -, -)
1	( $\infty$ , $\infty$ , -, -)	1	(1, 0, S, 0)
2	( $\infty$ , $\infty$ , -, -)	2	(1, 0, S, 0)
3	( $\infty$ , $\infty$ , -, -)	3	(2, 0, 2, 0)
4	( $\infty$ , $\infty$ , -, -)	4	(4, 0, 1, 0)
5	( $\infty$ , $\infty$ , -, -)	5	(5, 0, 2, 0)
6	( $\infty$ , $\infty$ , -, -)	6	(4, 0, 3, 0)
$\bar{S}$	( $\infty$ , $\infty$ , -, -)	$\bar{S}$	(6, 0, 4, 0)

<u>Iteration 2</u>	
a	$(\mu, \sigma^2, t, k)_{a,0}^j$
S	(0, 0, -, -)
1	(1, 0, S, 0)
2	(1, 0, S, 0)
3	(2, 0, 2, 0)
4	(4, 0, 1, 0)
5	(5, 0, 2, 0)
6	(4, 0, 3, 0)
$\bar{S}$	(6, 0, 4, 0)

**Table 4: Results of Algorithm for Example 2 - i = 1**

<b>a</b>	$(\mu, \sigma^2, t, k)_{a,1}^j$	<b>a</b>	$(\mu, \sigma^2, t, k)_{a,1}^j$
<b>S</b>	(0, 0, -, -)	<b>S</b>	(0, 0, -, -)
<b>1</b>	( $\infty$ , $\infty$ , -, -)	<b>1</b>	(1/8, 1/64, S, 1)
<b>2</b>	( $\infty$ , $\infty$ , -, -)	<b>2</b>	(9/32, 13/4096, S, 1)
<b>3</b>	( $\infty$ , $\infty$ , -, -)	<b>3</b>	(41/32, 13/4096, 2, 0)
<b>4</b>	( $\infty$ , $\infty$ , -, -)	<b>4</b>	(12/5, 9/100, 1, 1)
<b>5</b>	( $\infty$ , $\infty$ , -, -)	<b>5</b>	(11/5, 4/25, 2, 1)
<b>6</b>	( $\infty$ , $\infty$ , -, -)	<b>6</b>	(35/12, 5/144, 3, 1)
<b><math>\bar{S}</math></b>	( $\infty$ , $\infty$ , -, -)	<b><math>\bar{S}</math></b>	(25/6, 1/18, 4, 1)

<b>a</b>	$(\mu, \sigma^2, t, k)_{a,1}^j$
<b>S</b>	(0, 0, -, -)
<b>1</b>	(1/8, 1/64, S, 1)
<b>2</b>	(9/32, 13/4096, S, 1)
<b>3</b>	(41/32, 13/4096, S, 1)
<b>4</b>	(12/5, 9/100, 1, 1)
<b>5</b>	(11/5, 4/25, 2, 1)
<b>6</b>	(35/12, 5/144, 3, 1)
<b><math>\bar{S}</math></b>	(25/6, 1/18, 4, 1)

**Path is S , 1 , 4 ,  $\bar{S}$  with breakdown on (4,  $\bar{S}$ ) .**

**Effective length of path is approximately 4.9 .**

**True value is 4.5 .**

## LIST OF SYMBOLS

$(a, b)$	Arc joining $a$ and $b$
$l(a, b)$	Length of arc joining $a$ and $b$
$d(a, b)$	Deterministic decrease in length of $(a, b)$ resulting from a breakdown
$(D, t, k)_{a, i}$	A label for the deterministic case
$D_{a, i}$	First component of $(D, t, k)_{a, i}$
$t_{a, i}$	Second component of $(D, t, k)_{a, i}$
$k_{a, i}$	Third component of $(D, t, k)_{a, i}$
$L_{a, i}$	Length of the shortest $i$ -arc path from the source to node $a$
$\mu(a, b, i)$	Mean of the decrease in capacity of $(a, b)$ due to $i$ breakdowns
$\sigma^2(a, b, i)$	Variance of the decrease in capacity of $(a, b)$ due to $i$ breakdowns
$(\mu, \sigma^2, t, k)_{a, i}^j$	A label for the stochastic case
$\mu_{a, i}^j$	First component of $(\mu, \sigma^2, t, k)_{a, i}^j$
$(\sigma^2)_{a, i}^j$	Second component of $(\mu, \sigma^2, t, k)_{a, i}^j$
$t_{a, i}^j$	Third component of $(\mu, \sigma^2, t, k)_{a, i}^j$
$k_{a, i}^j$	Fourth component of $(\mu, \sigma^2, t, k)_{a, i}^j$
$[A, B]$	Cut set $[A, B]$
$V[A, B]$	Value of cut set $[A, B]$

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